- There are four hours available for the problems.
- Each problem is worth 10 points.
- Be clear when using a theorem; cite a source if necessary.
- Use a different sheet for each exercise
- Clearly write DRAFT on any draft page you hand in.

MOAWOA :: SOLUTIONS

28 April 2017

Problem 1. Given an integer $n \geq 3$, determine the smallest possible value of k such that $\mathbb{R}^k$ contains vectors $u_1, u_2, \ldots, u_n$ satisfying $u_i \cdot u_j = 0 \iff |i-j| > 1$.

Solution. We claim that $k = n-1$ is the smallest possible value. The example $u_1 = e_1$, $u_2 = e_1 + e_2$, $u_3 = e_2 + e_3$, $\ldots$, $u_{n-1} = e_{n-2} + e_{n-1}$, $u_n = e_{n-1}$, where $e_1, \ldots, e_{n-1}$ is the standard basis of $\mathbb{R}^{n-1}$, shows that such vectors indeed do exist for $k = n-1$. It remains to show that such vectors do not exist for $k < n-1$. We argue by contradiction. Suppose $u_1, \ldots, u_n \in \mathbb{R}^k$ satisfy the conditions of the problem, where $k < n-1$. The $n-1$ vectors $u_1, \ldots, u_{n-1}$ are linearly dependent over $\mathbb{R}^k$, so there exist $a_1, \ldots, a_{n-1} \in \mathbb{R}$, not all zero, such that $a_1u_1 + \ldots + a_{n-1}u_{n-1} = 0$. Let ℓ be the maximal index for which $a_\ell \neq 0$. Then $a_1u_1 + \ldots + a_\ell u_\ell = 0$, and taking the inner product with $u_{\ell+1}$ yields

$$\begin{aligned} 0 = 0 \cdot u_{\ell+1} &= (a_1u_1 + \ldots + a_\ell u_\ell) \cdot u_{\ell+1} \\ &= a_1(u_1 \cdot u_{\ell+1}) + \ldots + a_{\ell-1}(u_{\ell-1} \cdot u_{\ell+1}) + a_\ell(u_\ell \cdot u_{\ell+1}) \\ &= a_\ell(u_\ell \cdot u_{\ell+1}) \end{aligned}$$

because $u_i \cdot u_{\ell+1} = 0$ for $i < \ell$. From $a_\ell \neq 0$ it follows that $u_\ell \cdot u_{\ell+1} = 0$, contradiction. $\square$

Problem 2.

(a) A mathematician wants to tile his garden. He has enough tiles in $k \geq 2$ colours. He wants to put $n \geq 2$ tiles in a row, in such a way that no two tiles that are touching have the same colour. How many ways does this mathematician have to tile his garden?

(b) Another mathematician wants to put tiles around a circular pond in his garden. He needs $n \geq 2$ tiles and has $k \geq 2$ colours too. This time the first and the last tile touch and yet again no two touching tiles are allowed to have the same colour. How many ways does this mathematician have to tile his pond?

Solution.

(a) We let the mathematician put the tiles from the front to the back. For the first tile, the mathematician has k possibilities. For every tile after that he has $k-1$ possibilities, because he cannot choose the last colour that he used another time. So the number of tilings equals $k \cdot (k-1)^{n-1}$.

(b) For $n = 2$ and $n = 3$, the mathematician can choose n different colours and put the tiles in these colours behind eachother. For $n = 2$ there are $k(k-1)$ tilings and for $n = 3$ there are $k(k-1)(k-2)$ tilings.

Now we will prove with induction that there are $(k-1)^n + (-1)^n \cdot (k-1)$ tilings possible. This has just been shown for $n = 2$ and $n = 3$.

Now suppose the induction hypothesis holds for all $n < \ell$. We will now prove the statement for $n = \ell$. For every tiling of the pond with exactly ℓ tiles,we look the the northernmost tile. We look at what happens when we remove this tile. Two things could happen.

- The remaining $\ell - 1$ tiles form a correct tiling (when put back together). Conversely, we can expand every tiling with $\ell - 1$ tiles by puting an extra tile between the two northernmost tiles; you can choose $k-2$ colours for this tile.

  Vanwege de inductiehypothese voor $n = \ell - 1$, zijn er dus $(k-1)^{\ell-1} + (-1)^{\ell-1}(k-1)$ betegelingen met $\ell - 1$ tegels. Er zijn dus $(k-2) \cdot ((k-1)^{\ell-1} + (-1)^{\ell-1}(k-1))$ betegelingen met ℓ tegels waarbij dit voorkomt.

- De overgebleven $\ell - 1$ tegels vormen geen geldige betegeling (als je ze weer tegen elkaar aanschuift). Dit gebeurt als de twee noordelijke tegels dezelfde kleur hebben. Door die twee tegels samen te voegen tot één tegel, krijgen we een geldige betegeling met $\ell - 2$ tegels. Omgekeerd kunnen we elke betegeling met $\ell - 2$ tegels uitbreiden door de noordelijke tegel in twee stukken te hakken en er een tegel in één van de overgebleven $k-1$ kleuren aan toe te voegen.

  Vanwege de inductiehypothese voor $n = \ell - 2$ zijn er $(k-1)^{\ell-2} + (-1)^{\ell-2}(k-1)$ betegelingen met $\ell - 2$ tegels. Er zijn dus $(k-1) \cdot ((k-1)^{\ell-2} + (-1)^{\ell-2}(k-1)_3$ betegelingen met ℓ tegels waarbij dit voorkomt.

In totaal zijn er dus

$$\begin{aligned}((k-1)-1) \cdot \big((k-1)^{\ell-1} + (-1)^{\ell-1}(k-1)\big) + (k-1) \cdot \big((k-1)^{\ell-2} + (-1)^{\ell-2}(k-1)\big)\\ = (k-1)^{\ell} - (k-1)^{\ell-1} - (-1)^{\ell}(k-1)(k-2) + (k-1)^{\ell-1} + (-1)^{\ell}(k-1)^2\\ = (k-1)^{\ell} + (-1)^{\ell}\left((k-1)^2 - (k-1)(k-2)\right) = (k-1)^{\ell} + (-1)^{\ell}(k-1)\end{aligned}$$

betegelingen met ℓ tegels. Hiermee is de inductie rond.

Problem 3. Let $(G, \times)$ be a group. For any two group homomorphisms $f, g : G \to G$ we define the maps $f \circ g : x \mapsto f(g(x))$ and $f \cdot g : x \mapsto f(x) \times g(x)$. Also, a homomorphism

$f : G \to G$ is called *remarkable* when $f \cdot f = f \circ f$.

(a) Proof the following: if G is abelian, then $f \cdot g$ and $f \circ g$ are homomorphisms for any two homomorfisms $f, g : G \to G$. Is the converse also true?

(b) Determine the the number of natural numbers $n \in \mathbb{Z}_{>0}$ such that $n \leq 50$ and there exist exactly two remarkable homomorphisms $C_n \to C_n$. Here C_n is the cyclic group of order n.

Solution.

(a) For all $x, y \in G$ with G abelian, we have

$$\begin{aligned}(f \circ g)(x \times y) &= f(g(x \times y)) = f(g(x) \times g(y)) \\ &= f(g(x)) \times f(g(y)) = (f \circ g)(x) \times (f \circ g)(y), \text{ en}\end{aligned}$$

$$\begin{aligned}(f \times g)(x \times y) &= f(x \times y) \times g(x \times y) = f(x) \times f(y) \times g(x) \times g(y) \\ &= f(x) \times g(x) \times f(y) \times g(y) = (f \times g)(x) \times (f \times g)(y).\end{aligned}$$

It follows that $f \circ g$ and $f \times g$ are homomorphisms.

For the converse, we look at $f = g = \mathrm{id}$. In this case, the statement says that $f \times g : x \mapsto x \times x$ is a homomorphism. Specifically, this means that

$$(f \cdot g)(x \times y) = f(x \times y) \times g(x \times y) = x \times y \times x \times y$$

has to be equal to

$$(f \cdot g)(x) \times (f \cdot g)(y) = f(x) \times g(x) \times f(y) \times g(y) = x \times x \times y \times y.$$

By multiplying on the left side with x^{-1} and on the right side with y^{-1} we find that $y \times x = x \times y$ for all $x, y \in G$. So the group G is abelian.

(b) First off all, we identify the group C_n with $(\mathbb{Z}/n\mathbb{Z}, +)$. Every homomorphism $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is uniquely determined by $f(1)$. Suppose $f(1) = a$. Then it follows that $(f \circ f)(1) = f(a) = a \cdot f(1) = a^2$ and $(f \cdot f)(a) = f(a) + f(a) = 2a$. So the two homomorphism $f \circ f$ en $f \times f$ are the same iff $a^2 = 2a \in \mathbb{Z}/n\mathbb{Z}$. We will now look for solutions of the equation $a^2 = 2a$ modulo n.

If $n = p^k$ with p an odd prime, then there are two solutions, $a = 0$ and $a = 2$. This is proven as follows. If a is divisible by p, then a^2 has more factors of p than $2a$ and the only possibility for equivalence modulo p^k is when $a = 0$. If a is invertible modulo p^k, we multiply with a^{-1} and we find $a = 2$.

If $n = 2^k$, so a power of 2, then the equation has a single solution, $a = 0$, if $k = 1$ and two solutions, $a = 0$ en $a = 2$, if $k = 2$. For $k \geq 3$ there are at least three solutions: $a = 0$, $a = 2$ and $a = 2^{k-1}$.

If $n = p_1^{e_1} \cdot \ldots \cdot p_\ell^{e_\ell}$ a composite number, then we know the number of solutions for $a^2 = 2a$ in $\mathbb{Z}/n\mathbb{Z}$ is the product of the number of solutions in $\mathbb{Z}/p_1^{e_1}\mathbb{Z}, \ldots,$ and $\mathbb{Z}/p_\ell^{e_\ell}$.

The only n for which there are two solutions, are $n = 4$ and n of the form $n = p^k$ and $n = 2 \cdot p^k$ with p an odd prime and $k > 0$. Up until 50, those are the numbers: 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 17, 18, 19, 22, 23, 25, 26, 27, 29, 31, 34, 37, 38, 41, 43, 46, 47, 49 and 50. That 28 possibilities.

Problem 4. Determine all nonnegative integers n for which there is a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ for which there are n reals r such that the equation $f'(x) = r$ has a real solution x, but the equation $\frac{f(y)-f(z)}{y-z} = r$ has no real solutions $y \neq z$.

Solution.

The only such integers are $n = 0, 1, 2$. We start by proving the following lemma.

Lemma. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function and r an internal point of $\mathrm{im}(f')$. Then the equation $\frac{f(y)-f(z)}{y-z} = r$ has a real solution $y \neq z$.

Proof. Replacing $f(x)$ by $f(x) - rx$ we may assume that $r = 0$. In this case, the required comes down to showing that f is not injective. Suppose that f were to be injective. We know that any injective continuous function $f : \mathbb{R} \to \mathbb{R}$ is monotone, so $f' \geq 0$ (if f is increasing) or $f' \leq 0$ (if f is decreasing), contradicting that 0 is an internal point of $\mathrm{im}(f')$. □

Now, since f' is continuous, $\mathrm{im}(f') = f'(\mathbb{R})$ is a connected subset of $\mathbb{R}$, hence an interval. Therefore, there are at most 2 points in $\mathrm{im}(f')$ that are not internal points, showing $n \leq 2$. Now we show that $n = 0, 1, 2$ are all possible.

For $n = 0$, take $f(x) = x^2$, such that $f'(x) = 2x$. Now any point in $\mathrm{im}(f') = \mathbb{R}$ is an internal point, so the above lemma shows that this is indeed an example for $n = 0$.

For $n = 1$, take $f(x) = x^3$, such that $f'(x) = 3x^2$. Now, the only point in $\mathrm{im}(f') = [0, \infty)$ that is not an internal point is 0, hence $r = 0$ is the only real possibly satisfying the condition in the problem. Since f is injective, we see that $r = 0$ indeed satisfies.

For $n = 2$, we take the function f defined by

$$f(x) = \begin{cases} -4/3 + e^{x+1} & \text{if } x \leq -1; \\ x^3/3 & \text{if } -1 \leq x \leq 1; \\ 4/3 - e^{1-x} & \text{if } x \geq 1. \end{cases}$$

It is easy to check that this function is continuously differentiable with derivative given by

$$f'(x) = \begin{cases} e^{x+1} & \text{if } x \leq -1; \\ x^2 & \text{if } -1 \leq x \leq 1; \\ e^{1-x} & \text{if } x \geq 1. \end{cases}$$

Now $\mathrm{im}(f') = [0,1]$, so $r = 0$ and $r = 1$ are the only reals possibly satisfying the required condition. Since f is strictly increasing on each of the intervals $(-\infty, -1]$, $[-1, 1]$ and $[1, \infty)$ it is easy to check that $r = 0$ indeed satisfies. Furthermore, suppose there are reals $y > z$ with $\frac{f(y)-f(z)}{y-z} = 1$. By the mean value theorem there is some $x \in (z, y)$ with $f'(x) = \frac{f(y)-f(z)}{y-z} = 1$, hence $-1 \in (z, y)$ or $1 \in (z, y)$. We consider the case $1 \in (z, y)$. Now,

$$f(y) - f(z) = f(y) - f(1) + f(1) - f(z) = f'(x_1)(y-1) + f'(x_2)(1-z),$$

for some reals $x_1 \in (1, y)$ and $x_2 \in (z, 1)$. Now, using $y - 1, 1 - z > 0$, $f'(x_2) \leq 1$ and $f'(x_1) < 1$, we find

$$f(y) - f(z) < (y-1) + (1-z) = y - z,$$

a contradiction. Hence also $r = 1$ satisfies, providing an example for $n = 2$. $\square$

Problem 5. Let $(f_n)_{n\geq 0}$ be a sequence of functions $f : \mathbb{C} \to \mathbb{C}$ such that $f_0(z) = 0 \neq f_1(z)$ for all $z \in \mathbb{C}$ and $2f_{n+2} + 2^{-n} z f_{n+1}(z) + f_n(z) = 0$ for all $n \geq 0$ and all $z \in \mathbb{C}$. Prove that any root of f_{2017} is real.

Solution.

We can rewrite this recursion as $-zf_n(z) = 2^n f_{n+1}(z) + 2^{n-1} f_{n-1}(z)$ for $n \geq 1$. These recursion relations for $1 \leq n \leq 2016$ can be conveniently written in vector form as:

$$-z \begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ \vdots \\ f_{2015}(z) \\ f_{2016}(z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 2^{2016} f_{2017}(z) \end{pmatrix} + \begin{pmatrix} 0 & 2 & & & & \\ 2 & 0 & 4 & & & \\ & 4 & 0 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & 0 & 2^{2015} \\ & & & & 2^{2015} & 0 \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ \vdots \\ f_{2015}(z) \\ f_{2016}(z) \end{pmatrix}$$

Any root z of f_{2017} corresponds to the eigenvalue $-z$ of the real symmetric matrix on the right hand side (here we are tacitlyusing that $f_1(z) \neq 0$) and therefor z must be real. $\square$

Problem 6. Let a_1, a_2, b_1, b_2 be positive numbers such that $\gcd(a_1, a_2) = \gcd(b_1, b_2) = 1$ en $a_1 \neq \pm a_2$. Let $(x_n)_{n\geq 0}$ be the sequence of numbers given by $x_n = b_2 a_1^n - b_1 a_2^n$. Suppose $x_n \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Define $S := \{p \text{ prime} : \exists n \text{ s.t. } p \mid x_n\}$. Prove that S is infinite.

Solution.

We will call a sequence $c_1, c_2, \ldots$ *p-bounded* if $\{\mathrm{ord}_p(c_i) : i \in \mathbb{Z}_{\geq 0}\}$ is a bounded set, where $\mathrm{ord}_p(c_i)$ equals the unigque non-negative number such that $p^{\mathrm{ord}_p(c_i)} \mid c_i$ and $p^{\mathrm{ord}_p(c_i)+1} \nmid c_i$.

Notice that $|x_n| \to \infty$ when $n \to \infty$ since $a_1 \neq \pm a_2$. Suppose S is finite. We will prove that there exists a subsequence $x_{n_1}, x_{n_2}, \ldots$ such that for every prime $p \in S$ the subsequence is p-bounded. As a consequence the subsequence is itself bounded (since S is finite), so we have a contradiction.

Let $T := \{p \in S : p \nmid a_1 a_2\}$ and define $l = \phi(\prod_{p \in T} p^{\mathrm{ord}_p(x_0)+1})$, with ϕ Euler's phi function. Notice that

$$x_n - x_0 = b_2(a_1^n - 1) - b_1(a_2^n - 1).$$

For every n that is a multiple of l it holds that $a_1^n \equiv a_2^n \equiv 1 \mod p^{\mathrm{ord}_p(x_0)+1}$ (since $p \nmid a_1 a_2$). For such n, $x_n - x_0$ is divisible by $p^{\mathrm{ord}_p(x_0)+1}$. Because x_0 is not, by definition, it follows that x_n and x_0 contain the same amount of factors of p, i.e. $\mathrm{ord}_p(x_n) = \mathrm{ord}_p(x_0)$. This shows that $(x_{ln})_{n \geq 0}$ is p-bounded for all $p \in T$.

Now look at a prime p in S that is a divisor of $a_1 a_2$ and assume w.l.o.g. that p is a divisor of a_1 (and consequently not of a_2). Choosing n large enough, it follows that $\mathrm{ord}_p(b_2 a_1^n) > \mathrm{ord}_p(b_1)$, so $\mathrm{ord}_p(x_n) = \mathrm{ord}_p(b_1)$. Because $x_n \neq 0$ for all n, it follows that $(x_n)_{n \geq 0}$ is p-bounded and so is the subsequence $(x_{ln})_{n \geq 0}$ specified above.